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Title: Topological Methods for Visualization

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Topological Methods for Visualization

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LA-UR-16-22262

About me

- PostDoc @ Los Alamos National Laboratory
  - Extreme scale data:
    - Analysis
    - Sampling
    - Compression
  - Applications:
    - Climate
    - Plasma physics
    - Cosmology

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https://datascience.lanl.gov/

Education

- B.Sc. Computer Science/Mathematics
  - Human Computer Interaction
  - Computer Graphics
- M.Sc. Computer Science/Biology
  - Scientific and Information Visualization
  - Medical Visualization
  - Computational Geometry
  - Computer Vision
  - Image Processing
- Ph.D. Computer Science
  - Discrete Geometry
  - Topology
  - Visualization

Beyond Science

- Roller Derby
  - Player
    - 3 years
    - 3 countries
  - Non-Skating Official
    - 3 years
    - 7 countries
  - Skating Official
    - < 1 year
  - Coach
    - < 1 year
- Teaching!
  - Hour of Code
  - Expanding Your Horizons
  - T.A. for various lectures
    - Geometry
    - Topology
    - Numerics
    - Programming
    - Project management
  - This class!

Lecture Outline

- Basic Topological Concepts
  - Topological Spaces
  - Homeomorphisms
  - Homotopy
  - Betti numbers (if there's time)
- Scalar Field Topology
  - Finding topological features
  - Scalar Field Visualization
- Vector Field Topology
  - Finding topological features
  - Vector Field Visualization

Introduction to Topology

What is topology?

What can change and what is constant?

Definitions

Oxford dictionary:

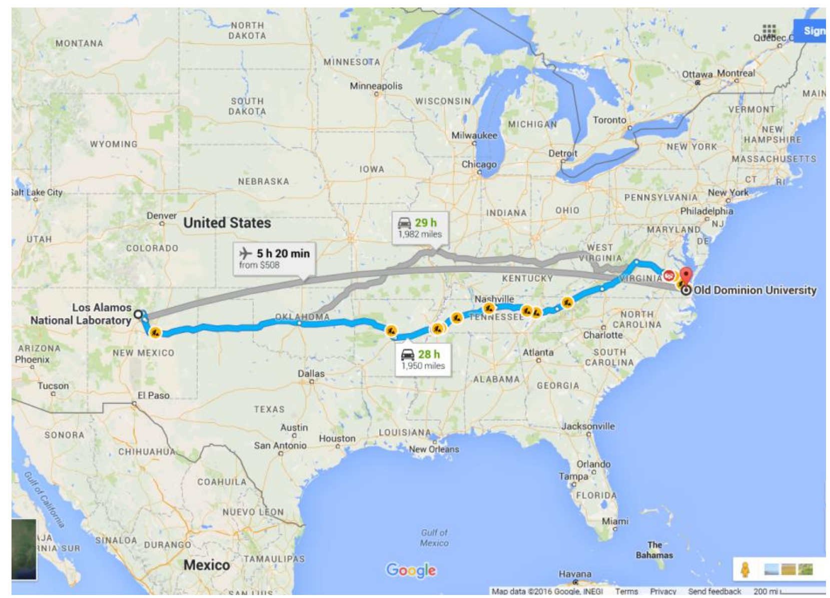
The study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.

Wiktionary:

[S]tudying those properties of a geometric figure or solid that are not changed by stretching, bending or similar homeomorphisms.

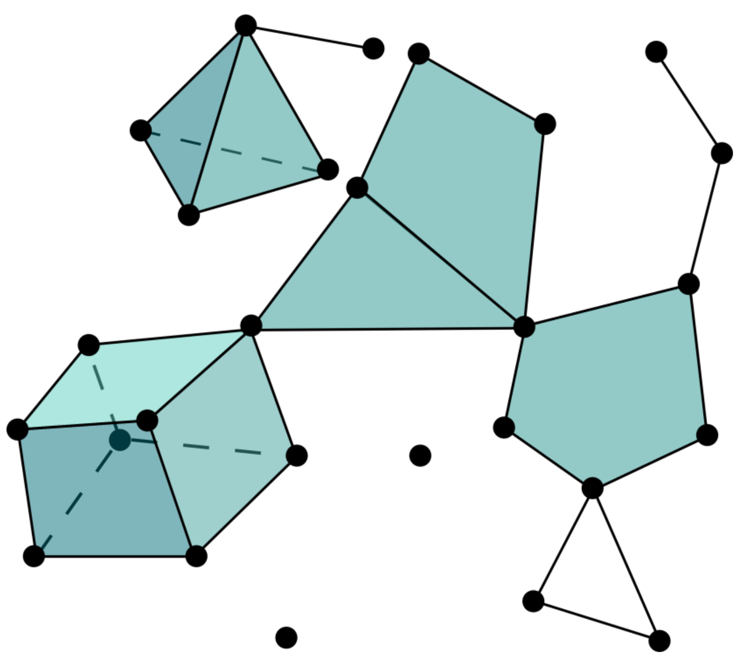
Why is it useful?

- Geographic Information Systems



Why is it useful?

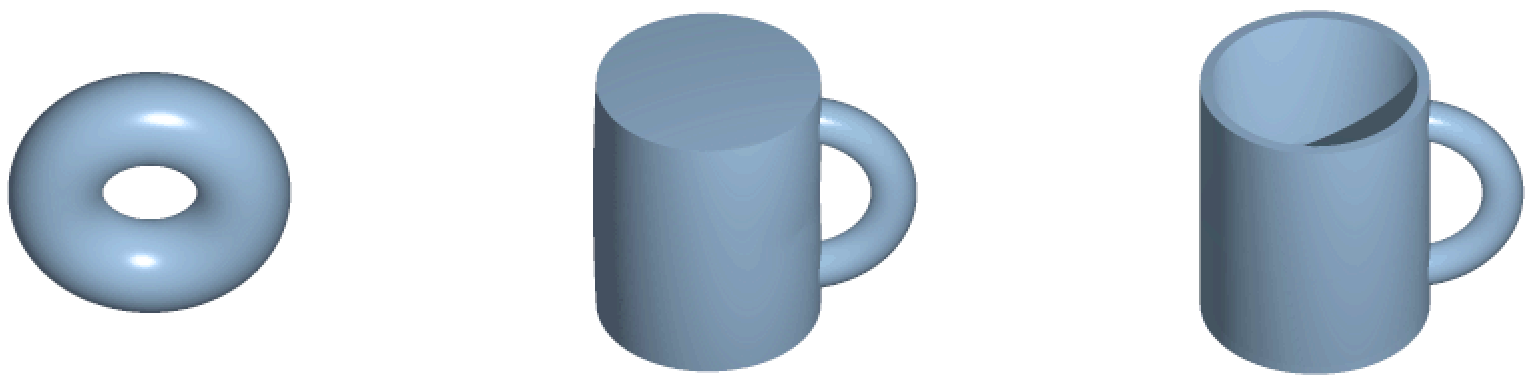
- Computer Graphics



- Robotics
  - Path planning
- Biology
  - Protein folding predictions
- …

Intuition

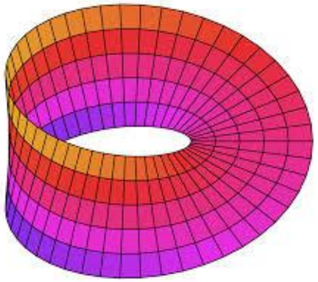
- Deform a geometric object continuously.
  - Pretend it’s made of infinitely stretchy material.
  - We cannot rip any part of the object.
  - Every point in the old object has to correspond to a point in the new object.



Animated Version

Make a call

Which of the following items are topologically identical to a donut?



Topological Spaces

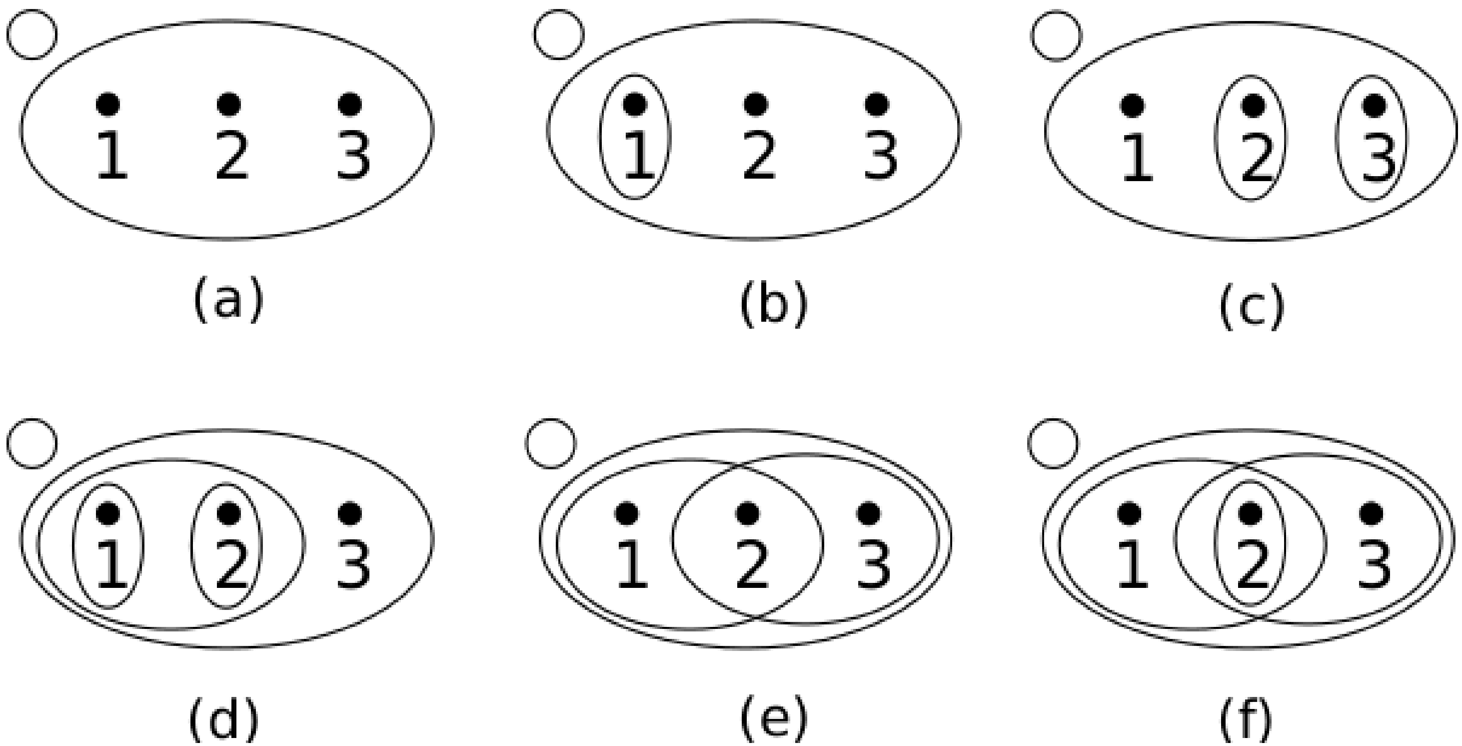
A topology on a point set X is a collection U of subsets of X, called open sets, such that

- X is open and the empty set $\emptyset$ is open
- If U_1 and U_2 are open, then $U_1 \cap \mathrm{U}_2$ is open
- If U_i is open for all i in some (possibly infinite, possibly uncountable) index set, then the union of all U_i is open.

The pair (X, U) is called a *topological space*.

Make a call

Which of the following are topological spaces?



Make a call

(a) Yes, this is the trivial topological space. It contains the whole set and the empty set.

(b) Yes.

(c) No, the union of {2} and {3} is missing.

(d) Yes.

(e) No, the intersection of {1,2} and {3,4} is missing.

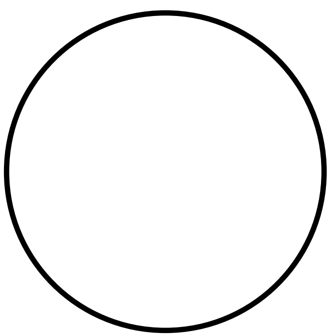
(f) Yes. The intersection missing in (e) is existent in this case.

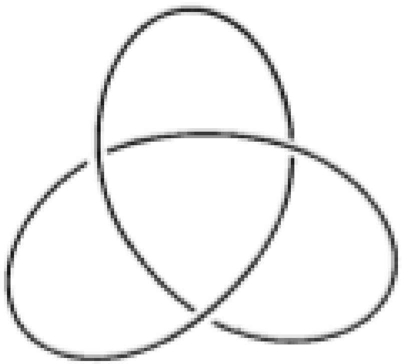
Topological Measures

- Homeomorphisms
- Homotopy
- Genus
- Betti numbers

Homeomorphism

If X and Y are topological spaces, a homeomorphism from X to Y is a bijective map $f: X \to Y$ such that f and f^{-1} are continuous, i.e. $U \subset X$ is open in X iff. $f(U)$ is open in Y.





Make a call

Which of the following letters are homeomorphic to each other?

A	B	C	D	E	F	G	H	I	J	K	L	M
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Make a call

- A R
- B
- C G I J L M N S U V W Z
- D O
- E F T Y
- H
- K X
- P Q

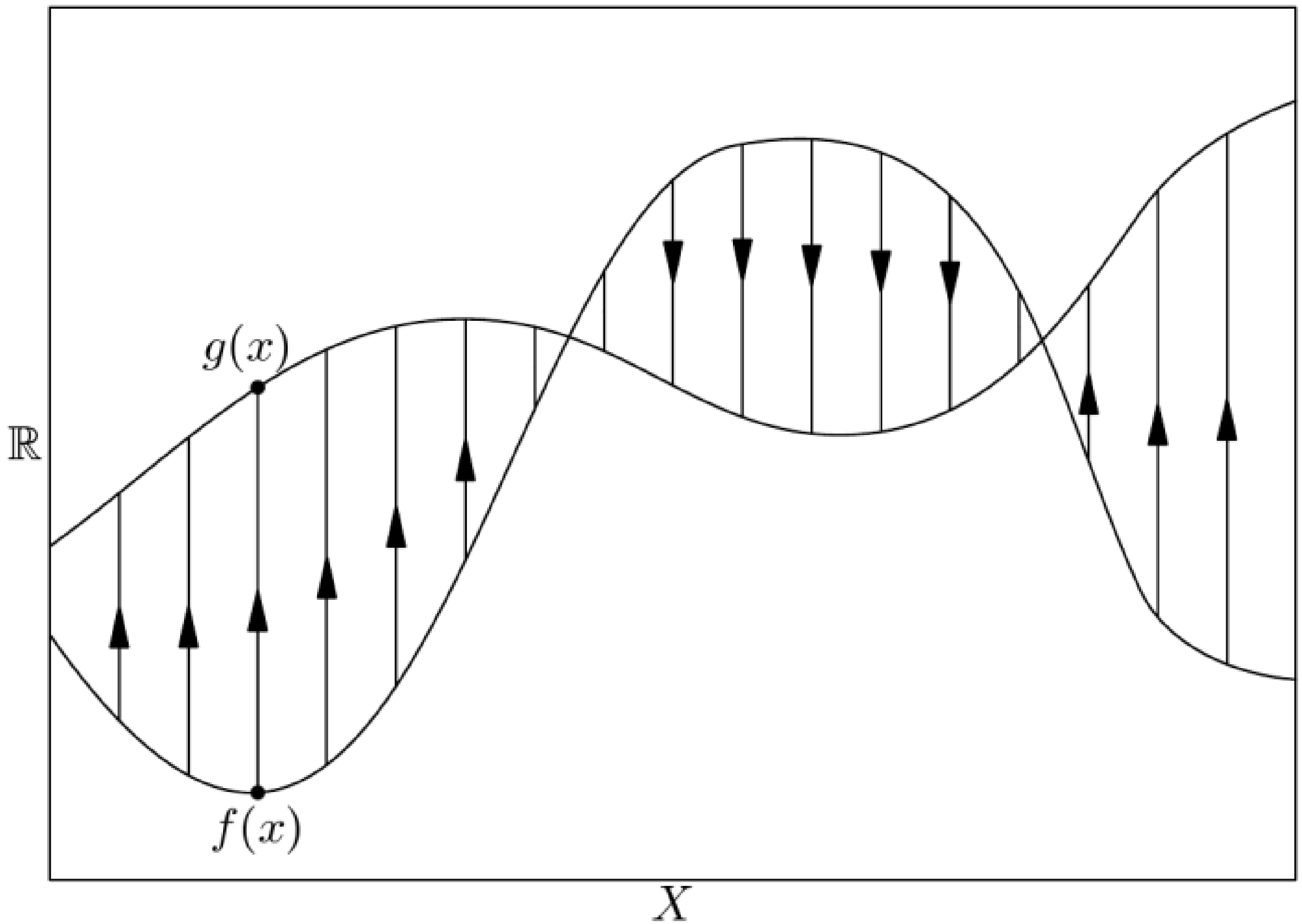
Homotopy

Two continuous maps $f, g: X \to Y$ between two topological spaces are called *homotopic* $(f \sim g)$ if there is a homotopy H between them. I.e. $H: X \times [0,1] \to Y$ with

$H(x, 0) = f(x)$, and

$H(x, 1) = g(x)$ for all x.

Homotopy



Make a call

Which of the following letters are homotopic to each other?

A	B	C	D	E	F	G	H	I	J	K	L	M
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

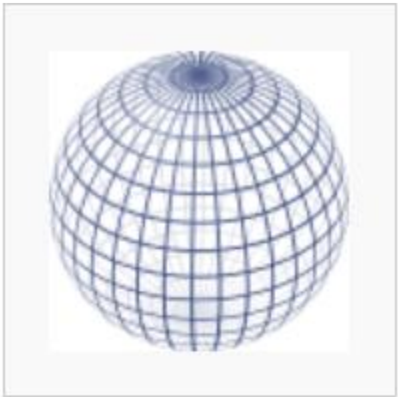
Make a call

- A D O P Q R
- B
- C E F G H I J K L M N S T U V W X Y Z

Genus

The *genus* of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. It is equal to the number of handles on it.

Genus



genus 0

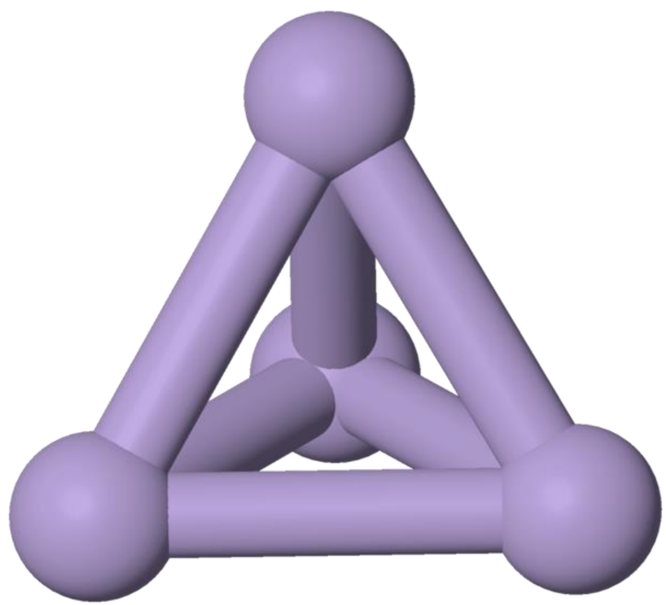
genus 1

genus 2

genus 3

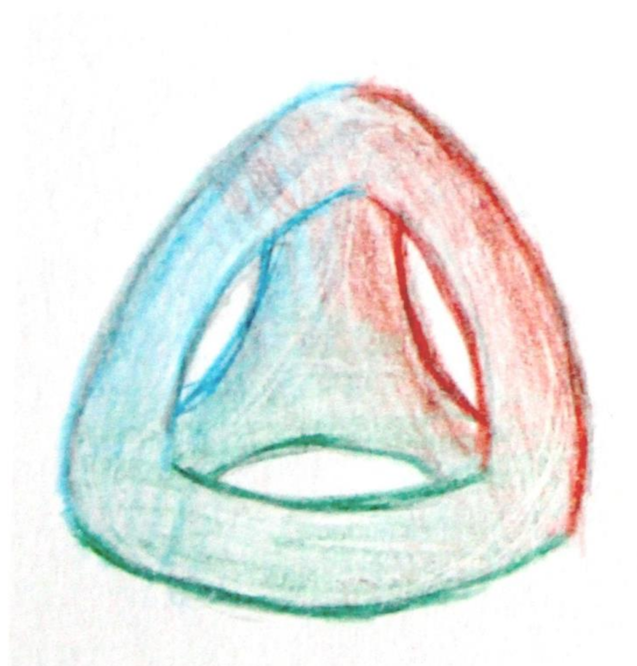
Make a call

What's the genus of a tetrahedron mesh (assuming the edges have mass)?



Make a call

A tetrahedron can be constructed from a 3-torus by transforming it slightly → genus is 3.



Betti Numbers

- Descriptor for topological properties
- Connectivity (or lack thereof) in different dimensions
- We'll look at Betti numbers 0, 1, and 2

Betti Numbers

- The zeroth Betti number counts connected components.
- The first Betti number counts the 2D generators for a object.
- The second Betti number counts 3D cavities inside the object.
- There are higher order Betti numbers than this.

Make a call

What are Betti numbers $\beta_0, \beta_1, \beta_2$ for these examples?

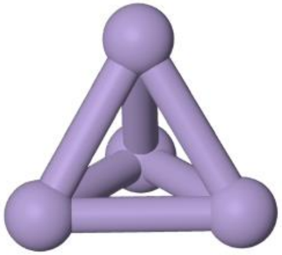
(a) Solid sphere.

(b) Surface of a sphere.

(c) Solid torus.

(d) Surface of a torus.

(e) Surface of a double torus.



(f) Surface of a tetrahedral wireframe.

Betti number β_0

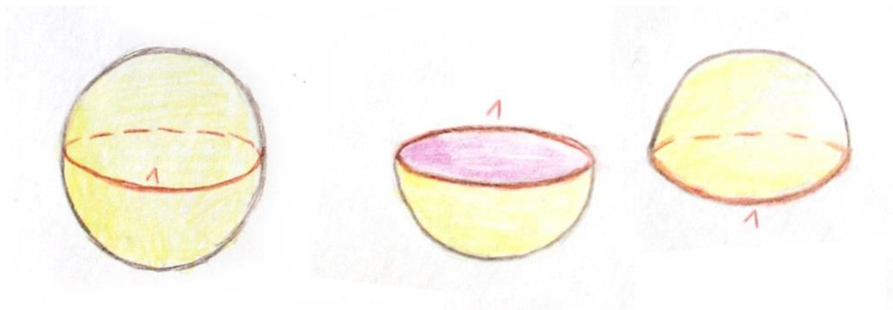
Each of the given object consists of one connected component since you can find a path from each point to any other. However, if you were to regard the falling sprinkles as part of the donut, you'd get 8 connected components for the donut.

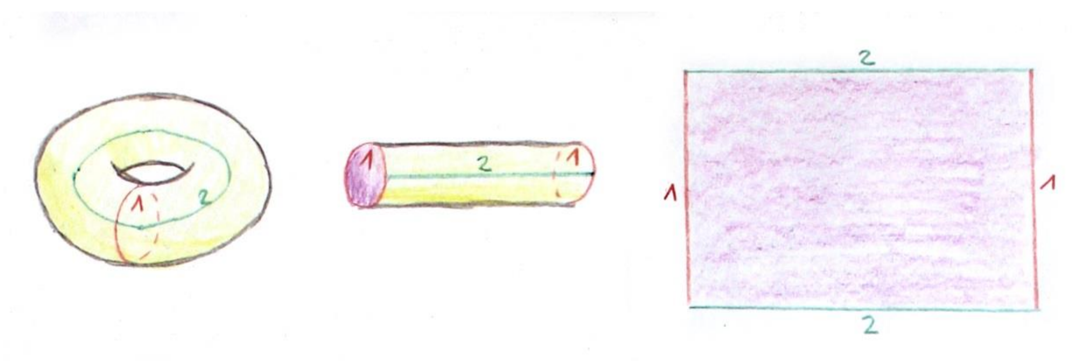
Betti number β_1

The number of 2D holes can be determined by counting the minimal number of cuts along closed curves that are needed to produce a surface that is flat (e.g. you could iron it easily). If you cut solid objects, you end up with slices that have a thickness (even if you cut them very thin).

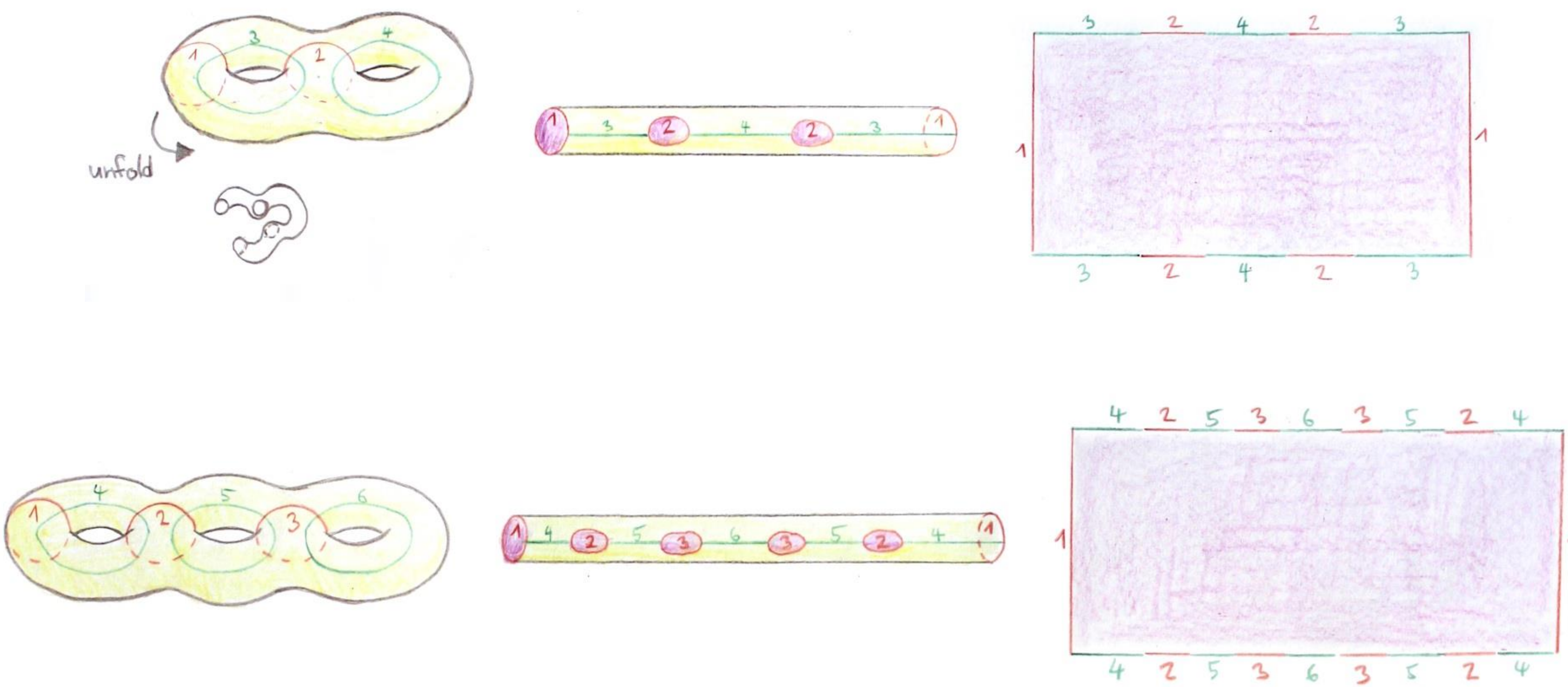
If you cut the surface of a sphere along any given curve, you end up with two connected components. For the torus and the double torus, you cut as indicated and end up with at surfaces. A tetrahedral frame is equivalent to a triple torus as we have seen in a previous exercise, so we can treat it as a triple torus.

Betti number β_1





Betti number β_1



Betti number β_2

Computing the number of 3D holes is simple. You simply imagine, how many valves you would need to inflate the object.

Solid objects cannot be inflated.

All the non-solid objects given here can be infated using just one valve. However, if the head rests on the double swim ring were separated from the ring parts, the Betti number would be bigger.

Scalar Field Topology

What are topological features?

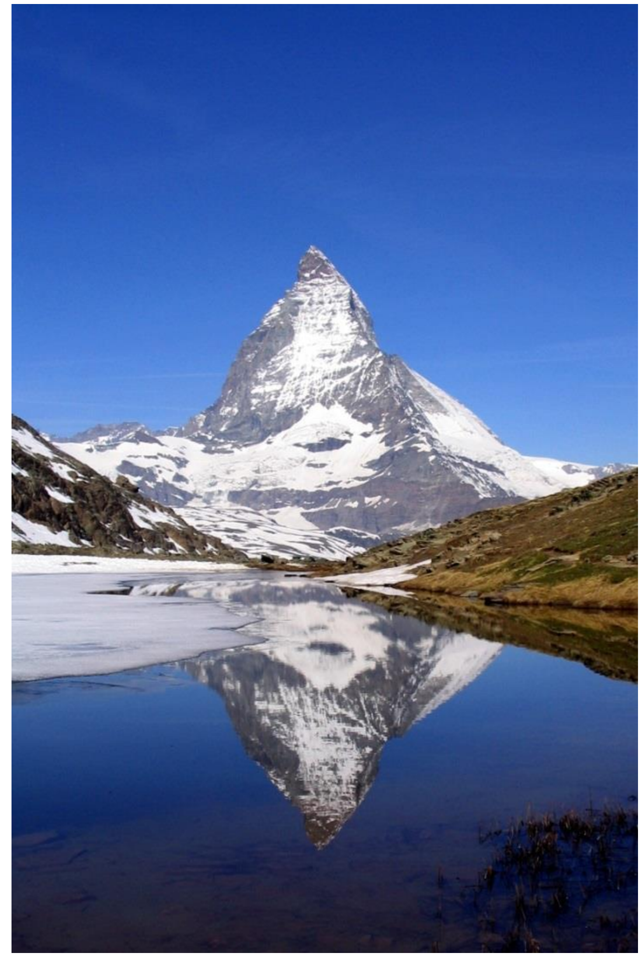
How can we determine them?

How can we visualize them?

Visualization

- Direct Volume Rendering
- Isolines/contours
- Contour Tree
- Reeb Graph

Scalar Field Features



How do we determine features?

Analytical description (1D/2D)

- First derivatives at critical points are zero
- Second derivatives are
  - Positive → min
  - Negative → max
  - Zero, and sign changes → saddle

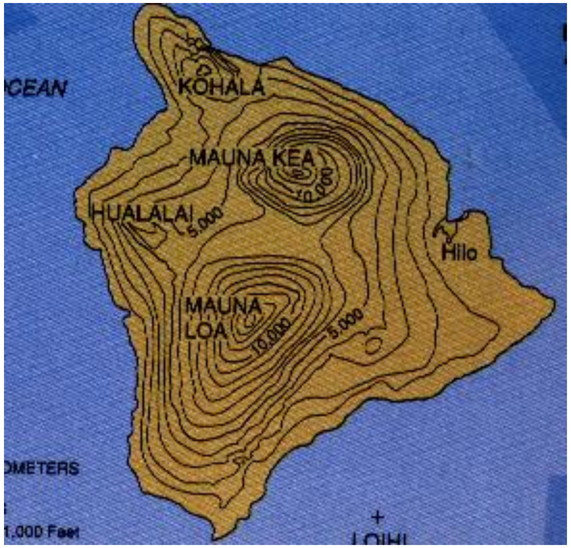
How do we determine features?

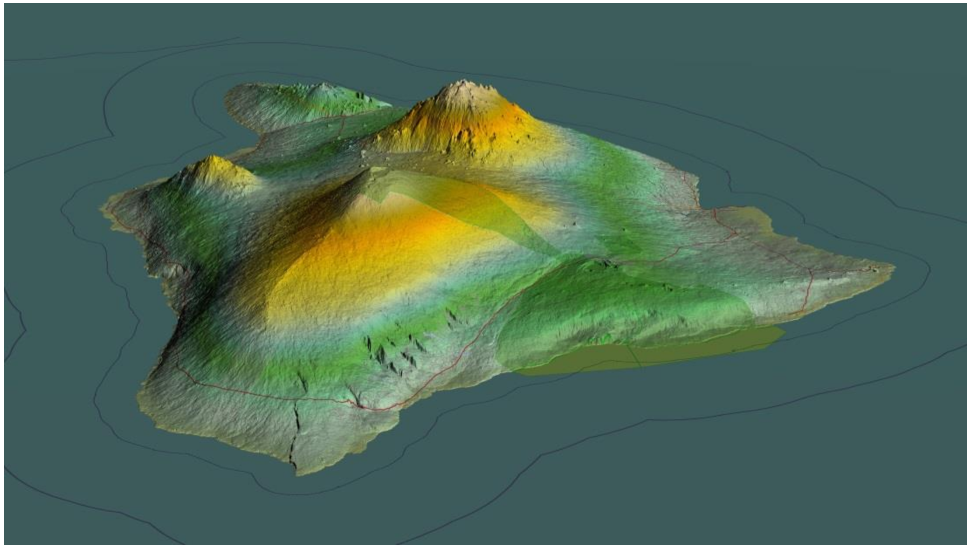
Discrete field (nD)

- Consider immediate neighbors
- Discrete derivatives
- Or just look at actual field
  - All smaller → max
  - All larger → min
  - Neighbors along one axis smaller, neighbors along other axis larger → saddle

Isolines/Contours

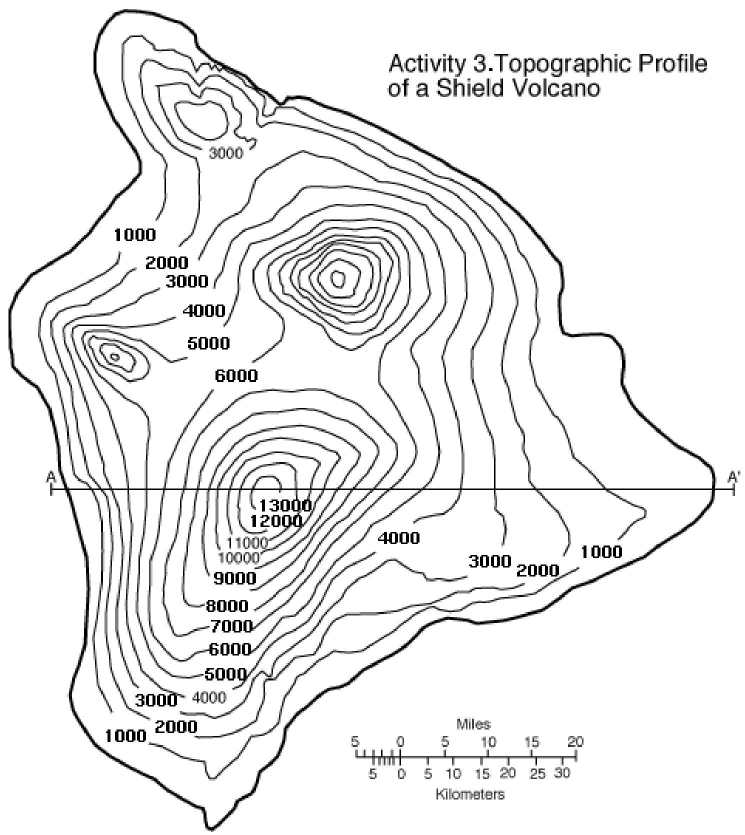
- Draw lines along all points with a particular value
  - Do this for a number of values for a better idea



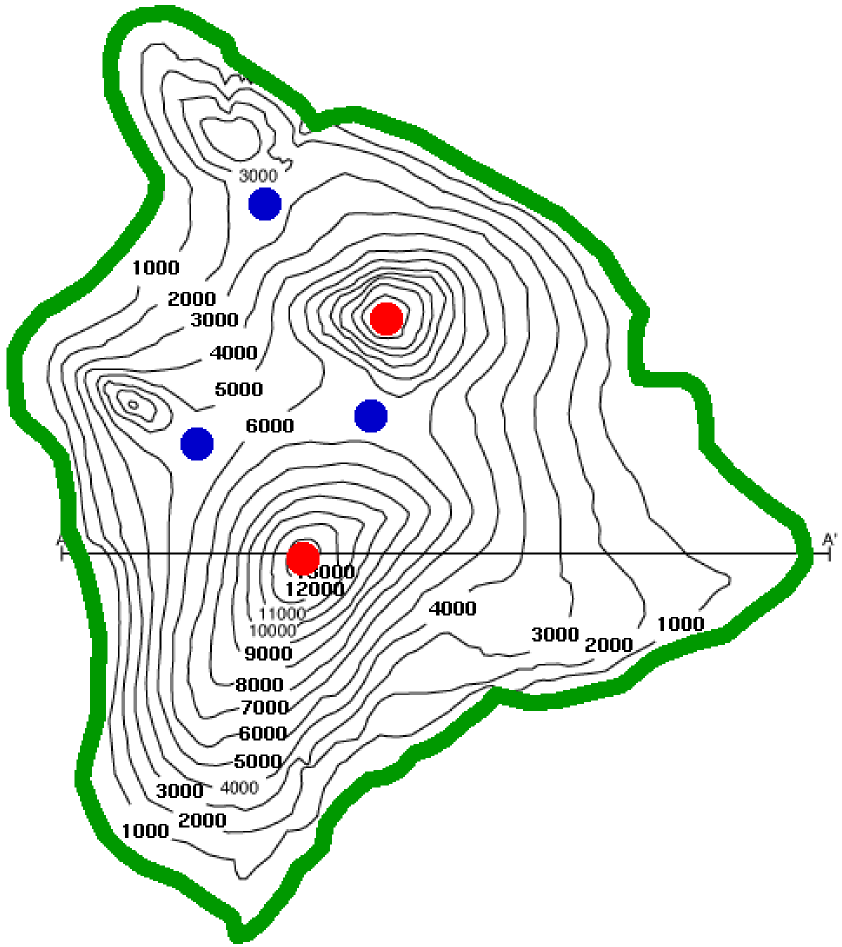


Make a call

How many min/max/saddles can you find?



Make a call



3000
1000
2000
3000
4000
5000
6000
12000
11000
10000
9000
8000
7000
6000
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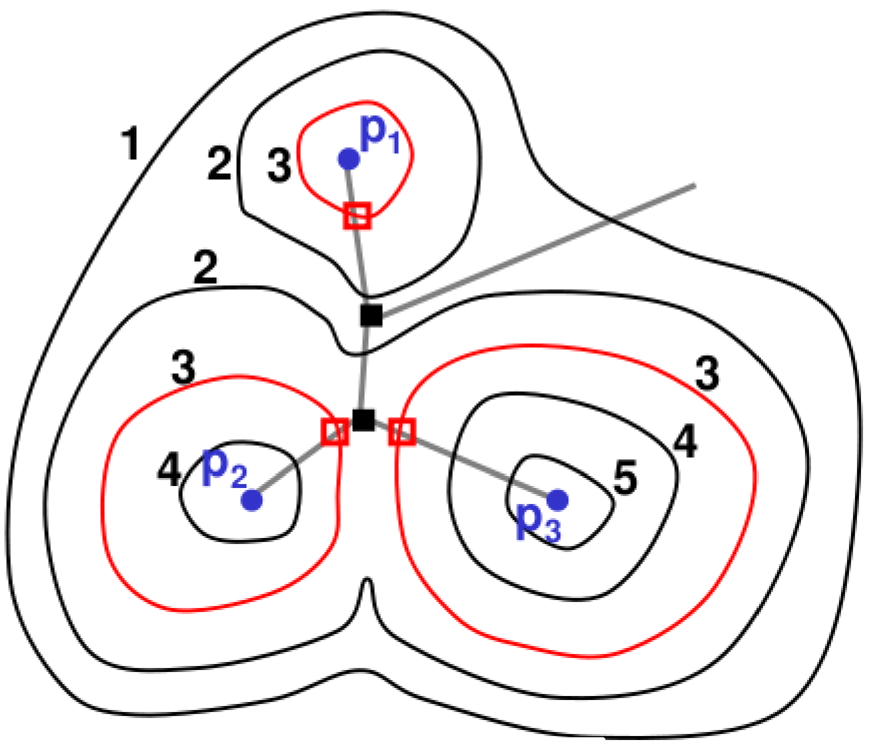
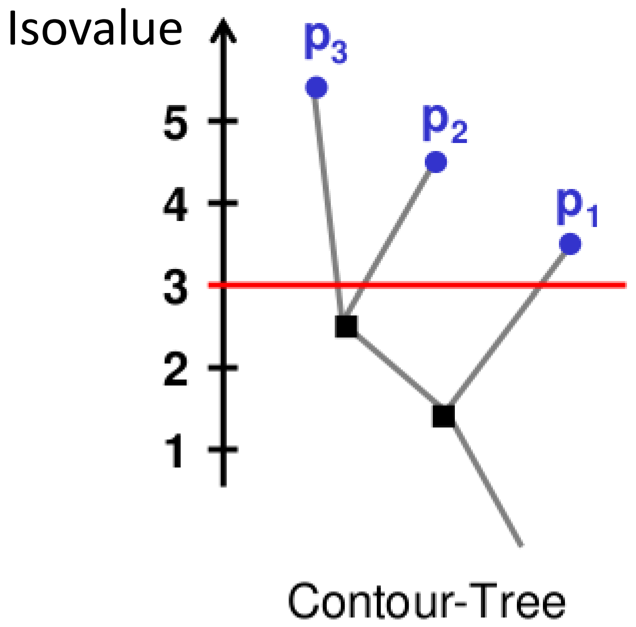
4000
3000
2000
1000
4000
3000
2000
1000
A
A'

Contour Trees

- Critical points are nodes
- Edges correspond to gradient towards next point
- Usually done to-scale, and as such not purely topological.
  - This makes it easy to determine, how many separate contour lines there should be for a given value.

Contour Trees





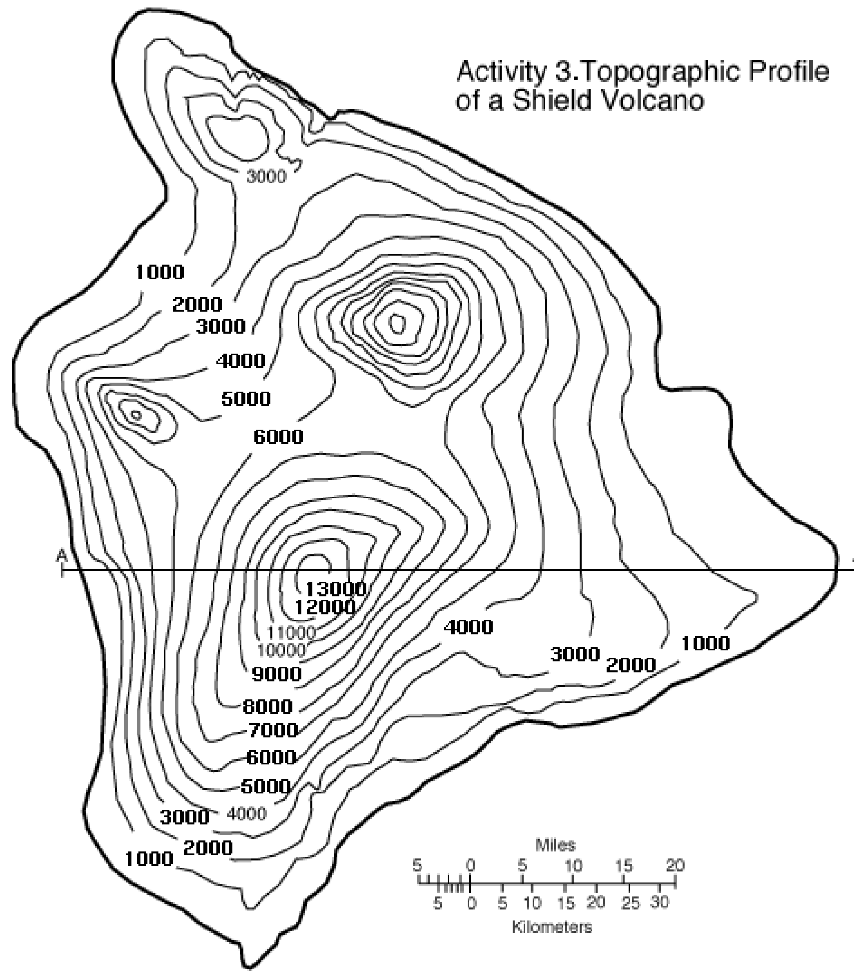
- Local extremum
- Saddle point
- Seed for isovalue 3

Make a call

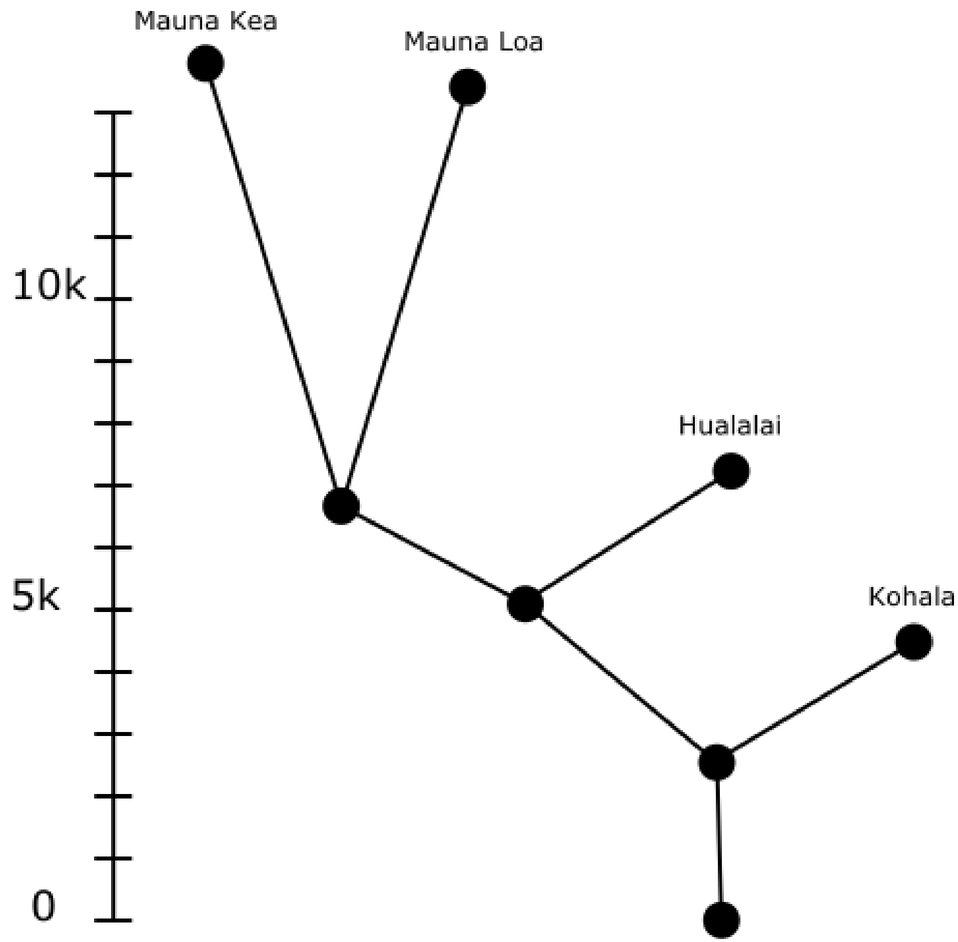
Draw the contour tree for this topographic map.

Elevation from top to bottom:

- Kohala: 5,479 ft
- Mauna Kea: 13,796 ft
- Hualālai: 8,271 ft
- Mauna Loa: 13,678 ft



Make a call

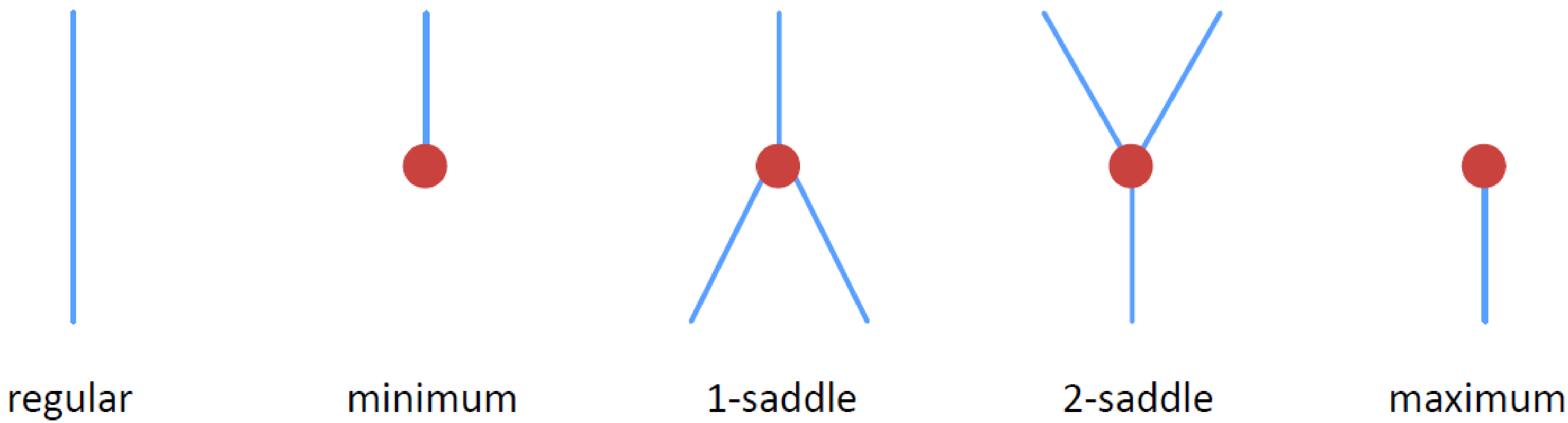


Reeb Graphs

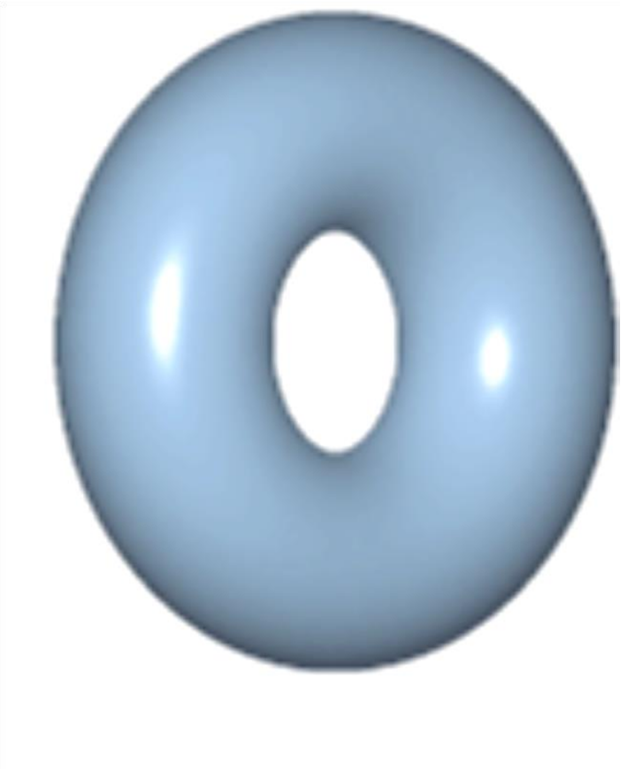
- Describe a solid in terms of connectivity
- Based on level sets
  - Imagine an area being flooded.
  - Everything submerged in water for a given water level is considered part of the corresponding level set.
- Reeb graphs give a topological representation of the progression between these level sets.

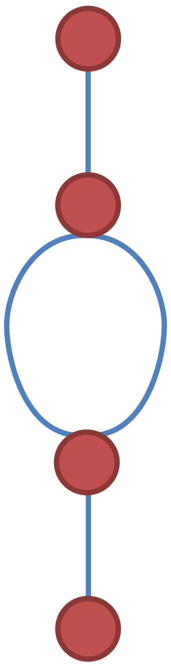
Reeb Graphs

There are different types of structures in the graph.



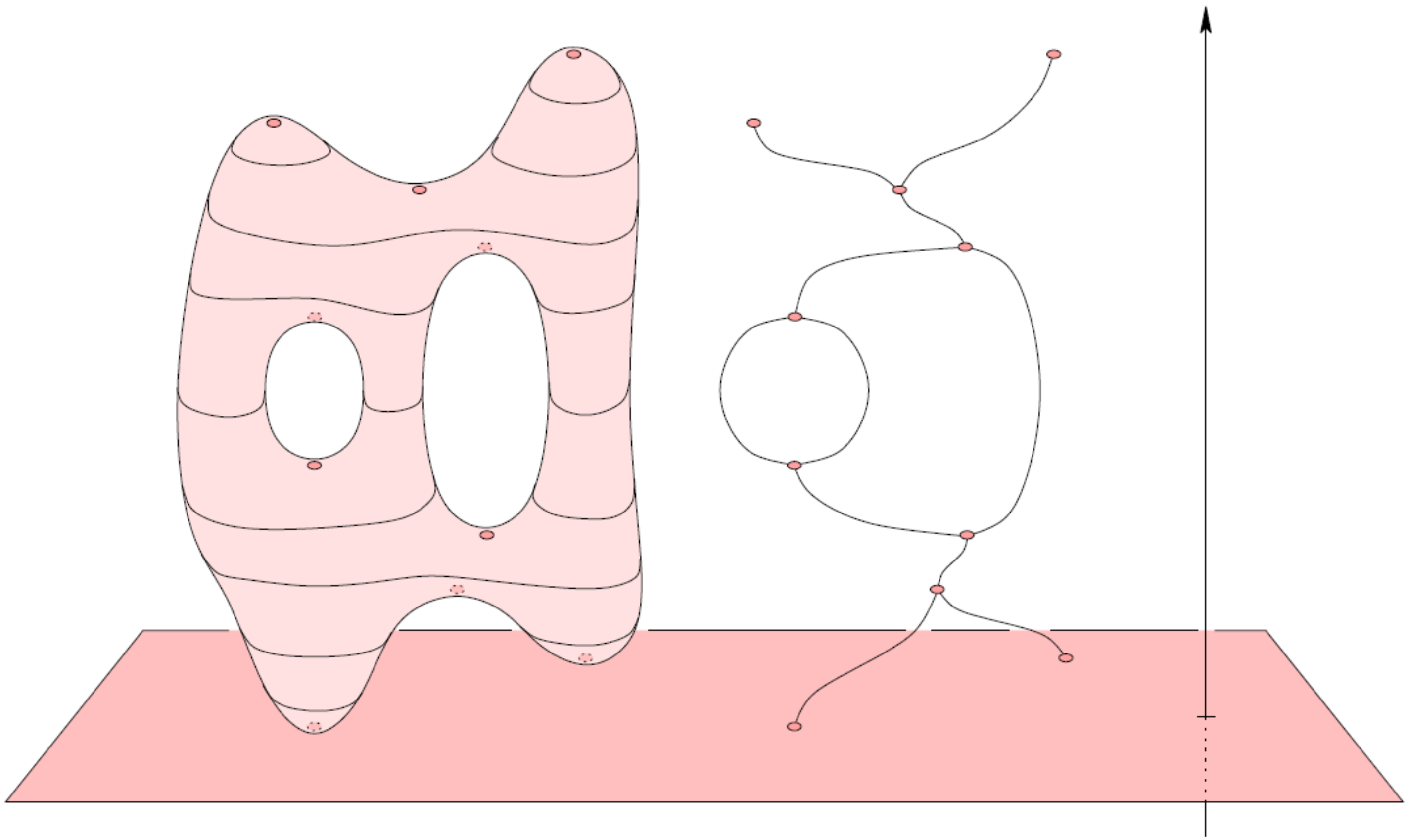
Reeb Graphs



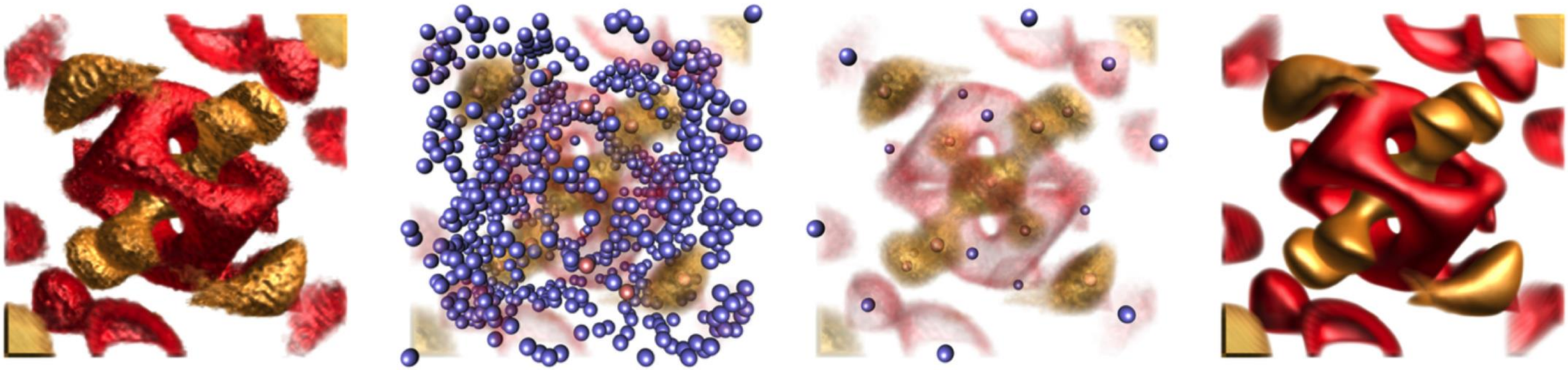


Make a call

Draw the Reeb Graph for this structure.

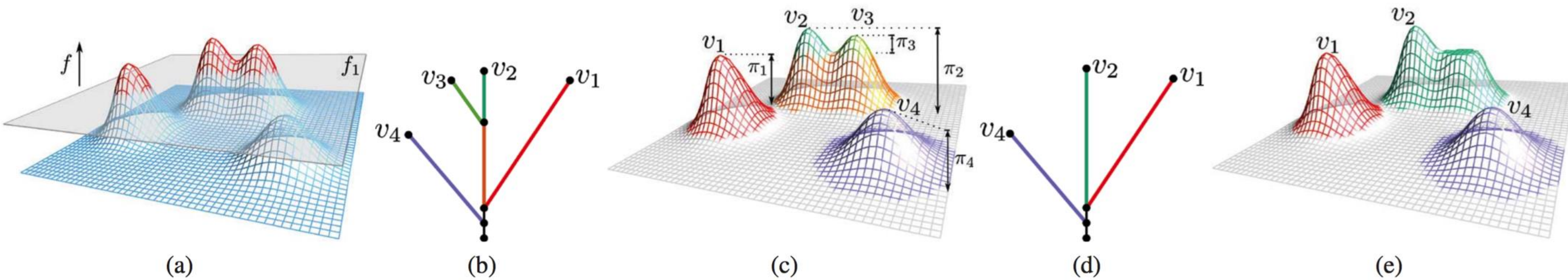


Denoising



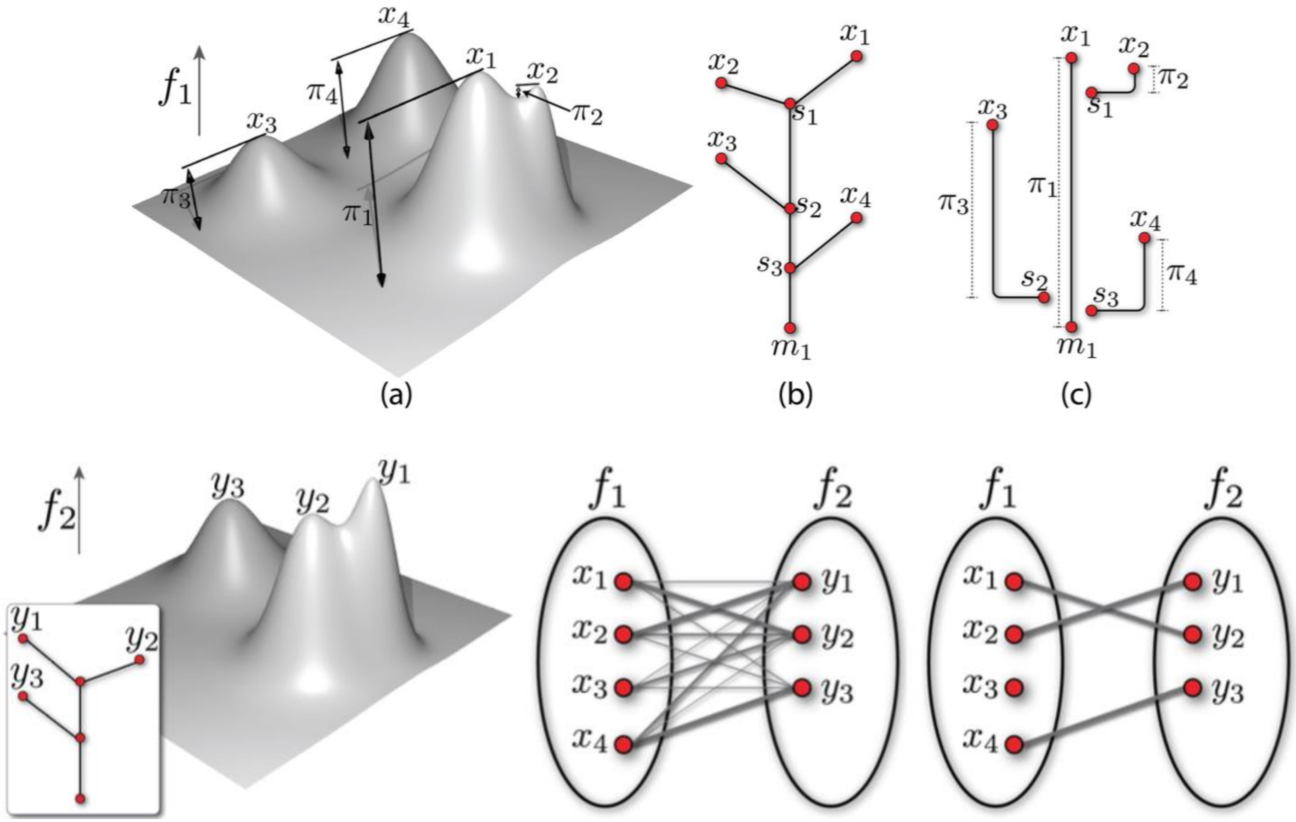
Guenther, David, et al. "Fast and Memory-Efficienty Topological Denoising of 2D and 3D Scalar Fields." *Visualization and Computer Graphics, IEEE Transactions on* 20.12 (2014): 2585-2594.

Contour Trees



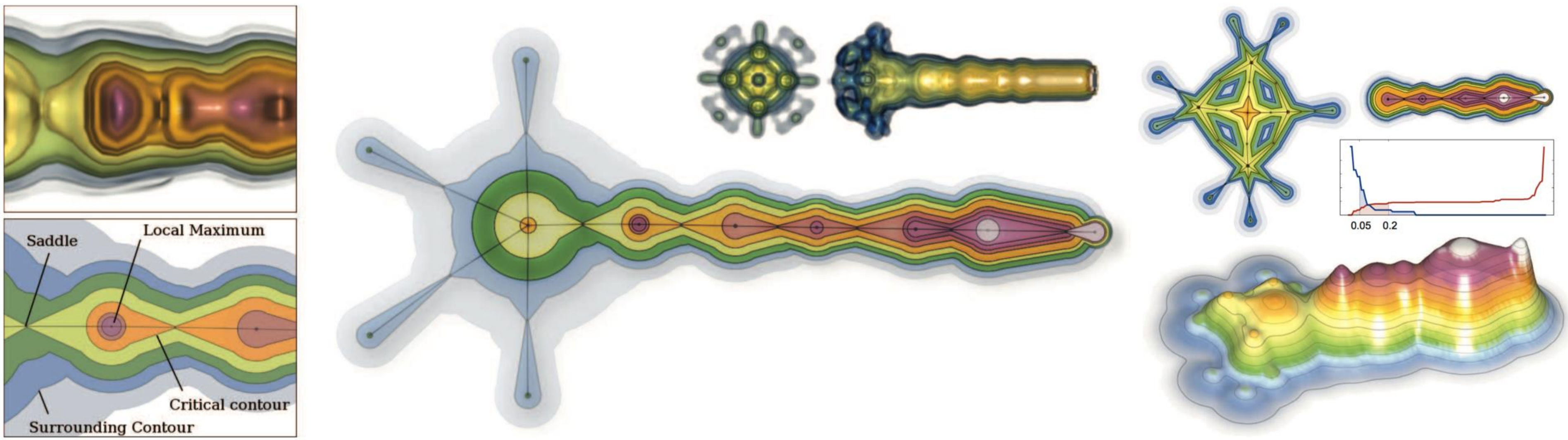
Doraiswamy, Harish, et al. "Using topological analysis to support event-guided exploration in urban data." *Visualization and Computer Graphics, IEEE Transactions on* 20.12 (2014): 2634-2643.

Contour Trees



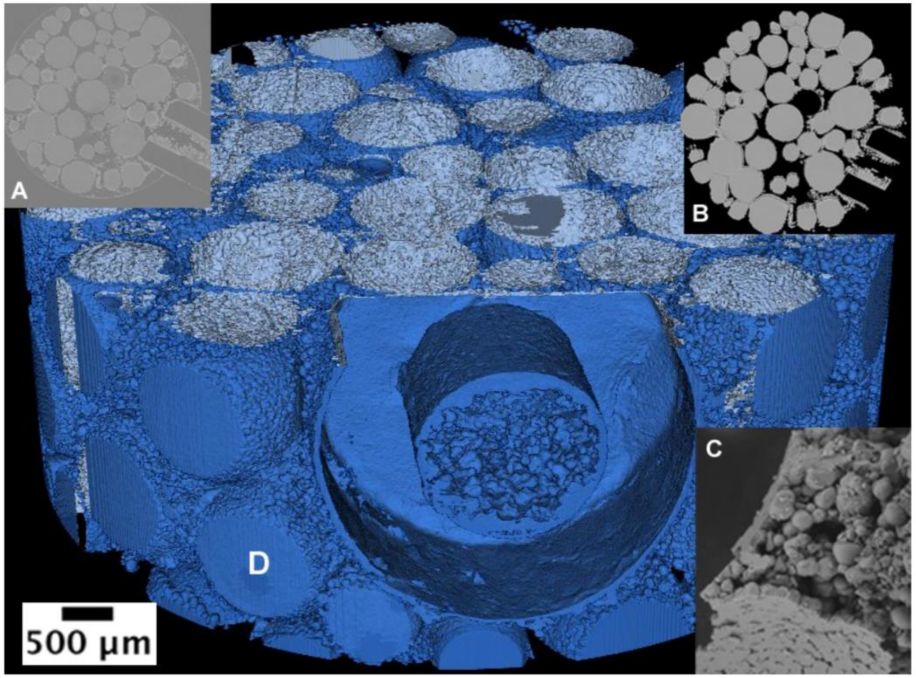
Poco, Jorge, et al. "Using Maximum Topology Matching to Explore Differences in Species Distribution Models." *Proc. IEEE SciVis* (2015).

Contour Trees



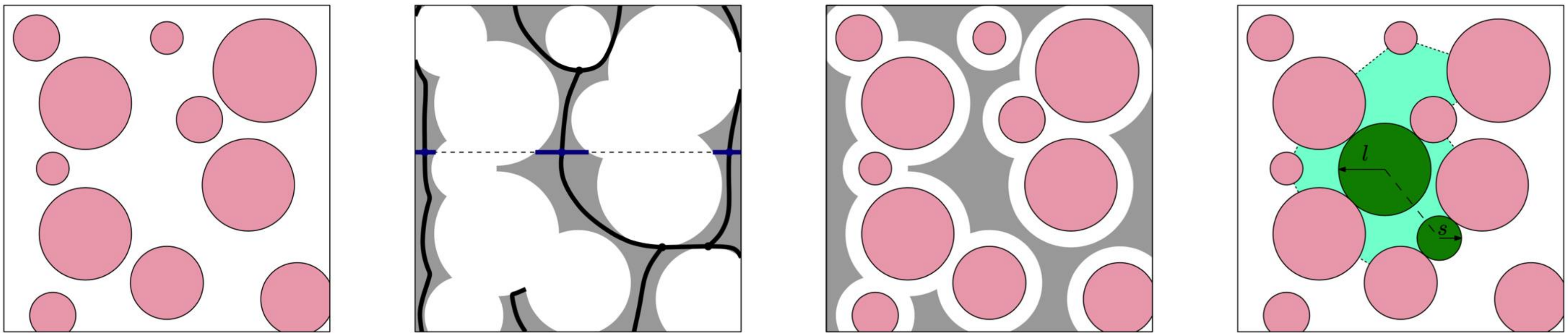
Correa, Carlos, Peter Lindstrom, and P-T. Bremer. "Topological spines: A structure-preserving visual representation of scalar fields." *Visualization and Computer Graphics, IEEE Transactions on* 17.12 (2011): 1842-1851.

Reeb Graphs

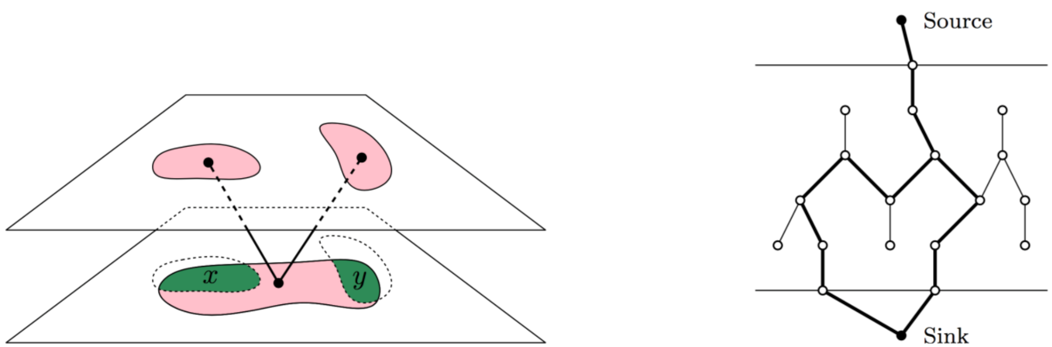


Ushizima, Daniela M., et al. "Augmented topological descriptors of pore networks for material science." *Visualization and Computer Graphics, IEEE Transactions on* 18.12 (2012): 2041-2050.

Reeb Graphs



(a) Void space (b) Reeb graph (c) Wide pathways (d) Pocket



(a) Edge capacities (b) Flow graph

Ushizima, Daniela M., et al. "Augmented topological descriptors of pore networks for material science." *Visualization and Computer Graphics, IEEE Transactions on* 18.12 (2012): 2041-2050.

Vector Field Topology

What are topological features?

How can we determine them?

How can we visualize them?

Visualization

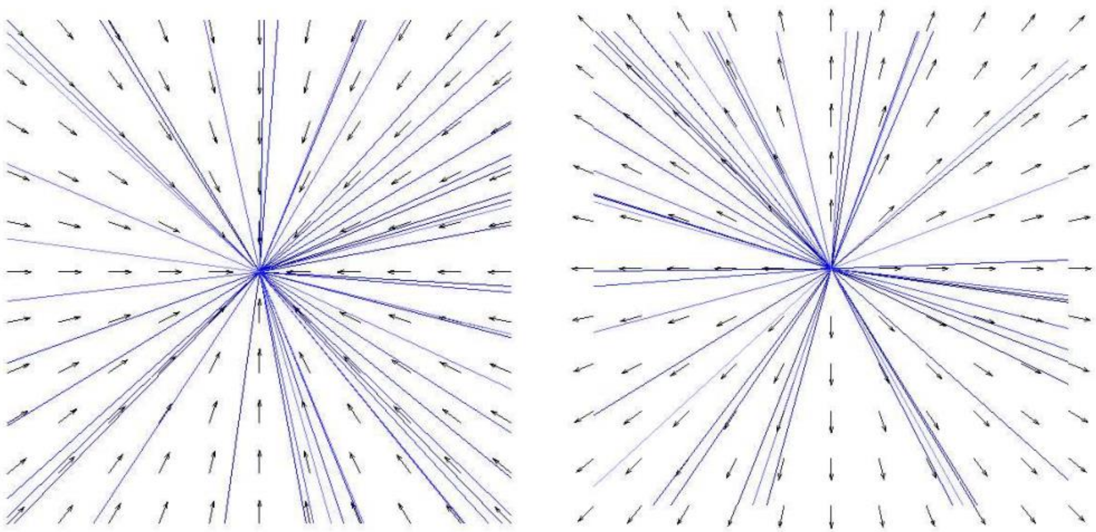
- Glyphs
- Line Integral Convolution
- Streamlines (or other flow vis techniques)
- Topological classification into similar regions

Features in Vector Fields

Look at the neighborhood of a point

- What happens with respect to the point
- Is the point located at a discontinuity?
  - Then it is a *critical point*.
  - Derivative at this point is zero.
- Separatrices are lines that are tangential to the vector field and that connect critical points
- Vortex cores trace the center of a vortex in a vector field

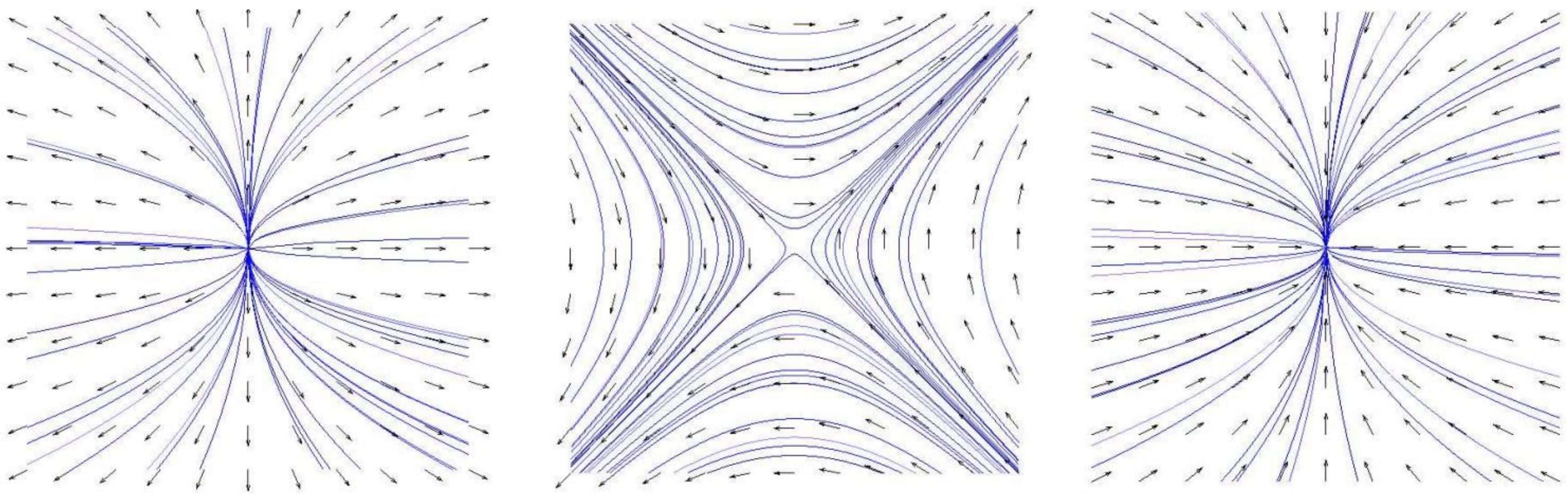
Critical Points



(a) attracting star (b) repelling star

Figure 3.1: Stars

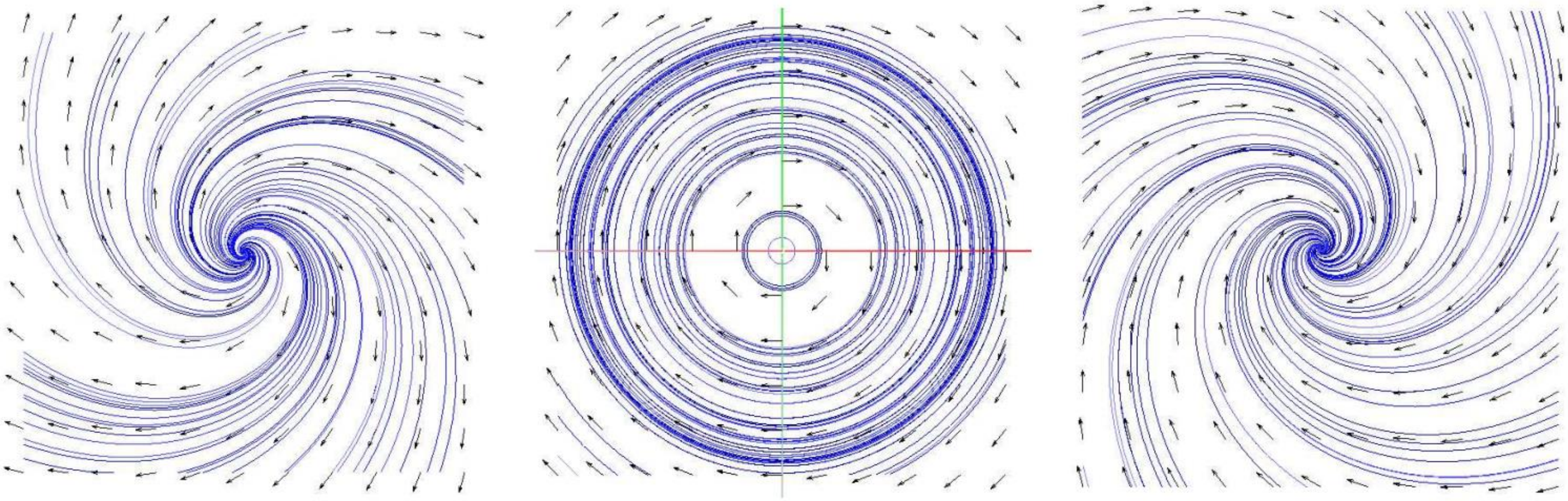
Critical Points



(a) repelling node (b) saddle (c) attracting node

Figure 3.2: Nodes and saddle

Critical Points



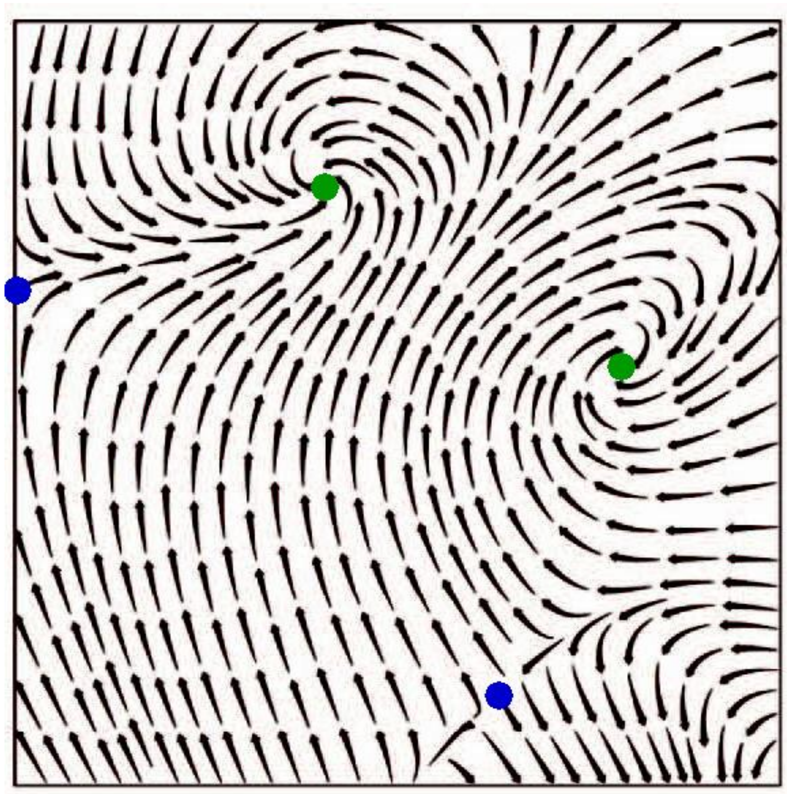
(a) repelling spiral (b) cycle (c) attracting spiral

Figure 3.3: Spirals and cycle

Make a call

Which critical points can you see in this field?

Make a call

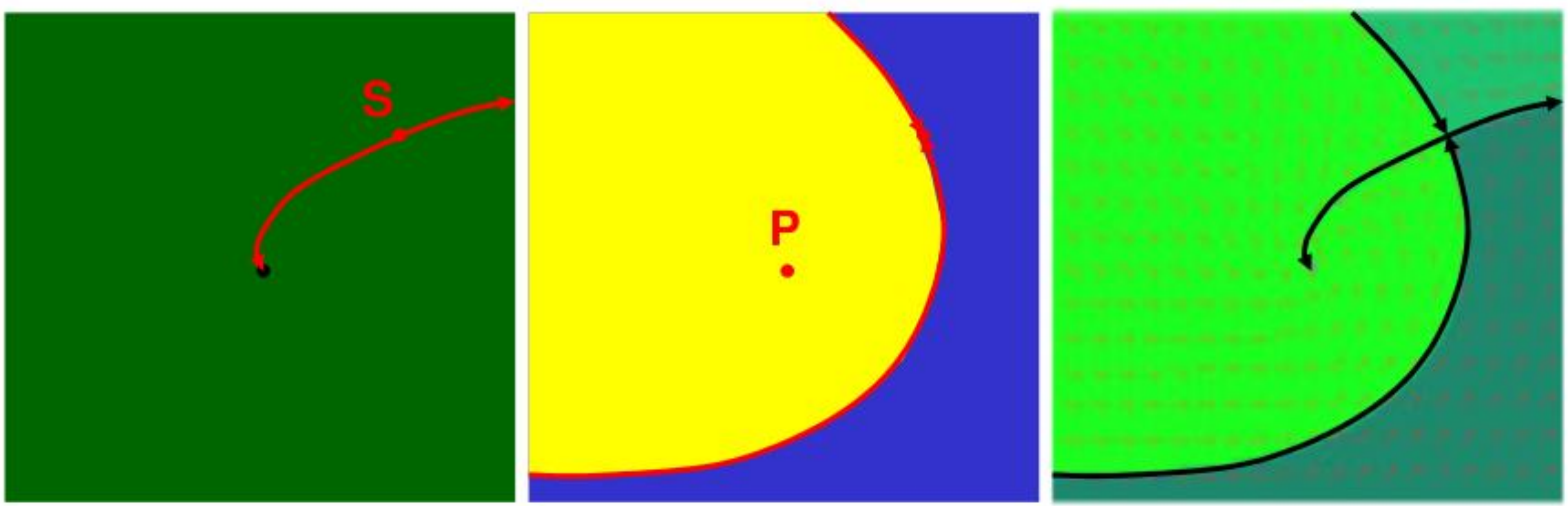


Detecting Features

- Compute Jacobi Matrix of the field
- Use Jordan Normal Form to obtain eigenvalues
  - $\lambda_1, \lambda_2 > 0$: Repelling node
  - $\lambda_1, \lambda_2 < 0$: Attracting node
  - $\lambda_2 < 0 < \lambda_1$: Saddle
  - Complex eigenvalues
    - $Re(\lambda)$ determines repellor (>0)/attractor (<0)/circle (0)
    - $Im(\lambda)$ determines rotating direction

Topology of a Vector Field

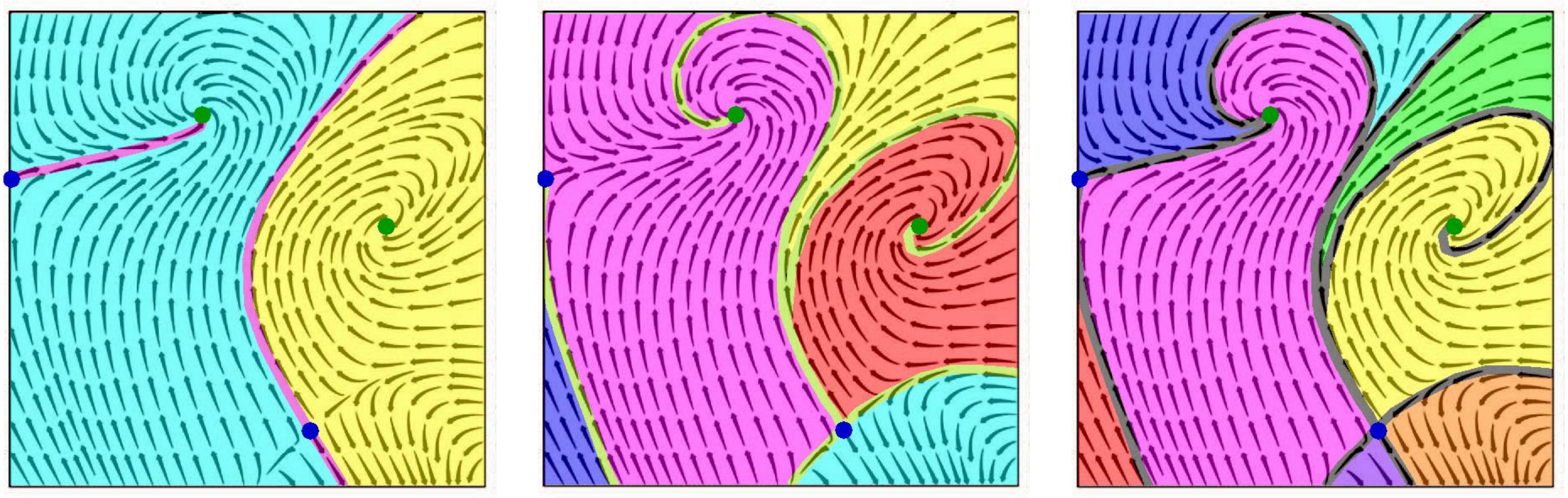
- α sets contain all streamline points that originated from the same critical point
- ω sets contain all streamline points that end in the same critical point



Make a call

Draw a sketch of the alpha set, omega set, and topological representation of the field.

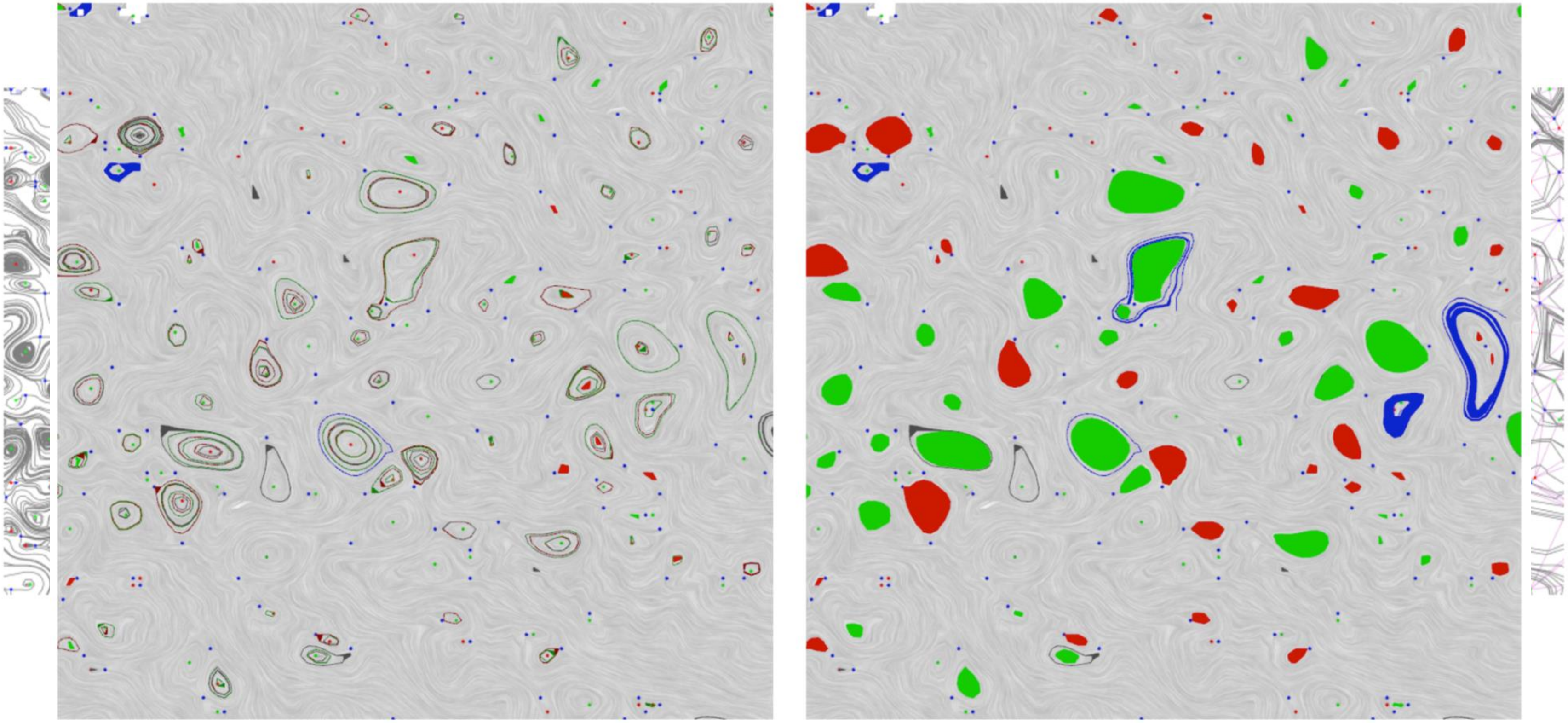
Make a call



Time-Dependent Fields

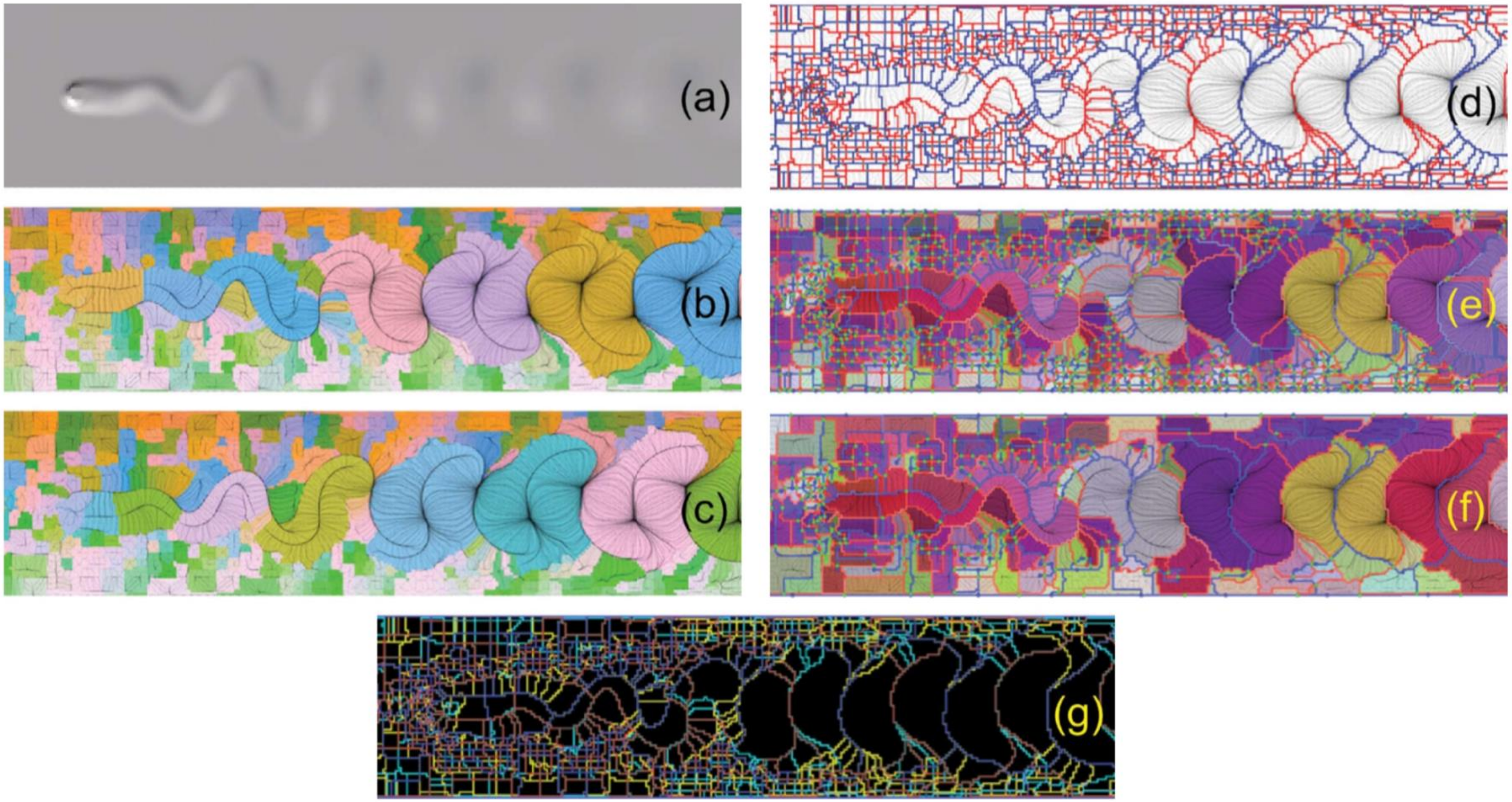
- Birth/death of features
- Entry/exit from domain
- Split/merge of features

Examples



Szymczak, Andrzej, and Levente Sipeki. "Visualization of Morse connection graphs for topologically rich 2D vector fields." *Visualization and Computer Graphics, IEEE Transactions on* 19.12 (2013): 2763-2772.

Vector Field Topology



Gyulassy, Attila, et al. "Conforming Morse-Smale Complexes." *Visualization and Computer Graphics, IEEE Transactions on* 20.12 (2014): 2595-2603.

Further Reading

- Edelsbrunner, Herbert, and John Harer. *Computational topology: an introduction.* American Mathematical Soc., 2010.
- Jaenich, Klaus. *Topology (Undergraduate Texts in Mathematics).* (1984).
- Lee, John. *Introduction to topological manifolds*. Vol. 940. Springer Science & Business Media, 2010.